6. Internal Loadings in Structural Members

When we determined forces in structural components in previous chapters, we were determining the net force acting in the component. However, in order to properly design structural components, we must know the distribution of these forces within the component.

For example, common sense indicates that if a concrete beam bends under a downward load, the fibers in the top of the beam will be in compression; whereas the fibers in the bottom of the beam will be in tension. We also know that concrete is much stronger in compression than it is in tension. In fact, we include steel reinforcing bars in concrete beams because of the ability of steel to resist tension. We need to know how tension/compression are distributed across the beam cross section so that we can select the amount of steel required and determine its optimum location.

In this chapter, we will determine the normal force, shear, and moment at a point in a structural component. The actual distribution of the forces over the cross section of the component will be dealt with in more advanced design courses.

6.1 - Internal loads at a point within a structural component. For a coplanar structure, the internal load at a specified point will consist of a normal force, **N**, a shear force, **V**, and a bending moment, **M**. Fig. 6.1.1 shows the sign convention to be employed.

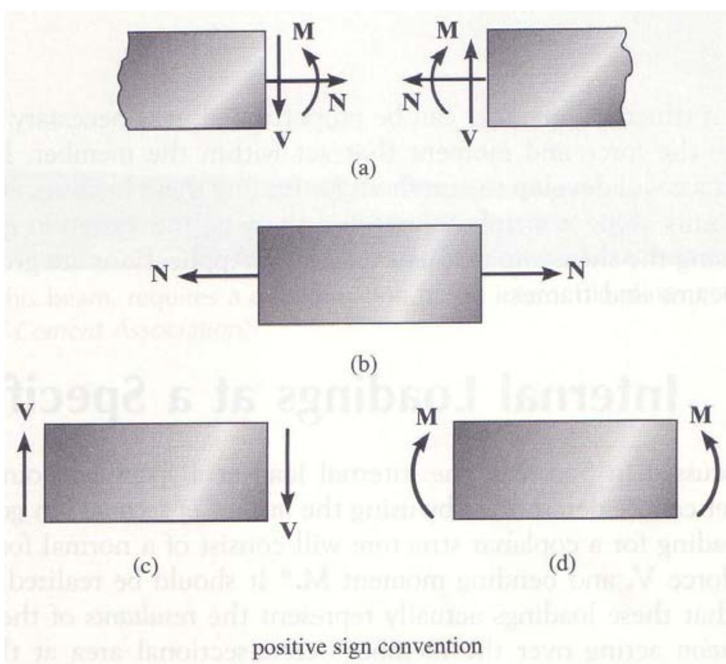


Fig. 6.1.1 - Sign convention for internal positive forces.

(1) Positive shear, **V**, tends to rotate the component clockwise. Note that the shear is in opposite directions on either side of a cut through the component; nevertheless, each of the two shear components tends to rotate its respective section clockwise. Therefore, each is positive.

(2) Positive normal force, **N**, tends to elongate the components. Again, note that the normal forces act in opposite directions on either side of the cut; nevertheless, each of the two normal components tends to elongate its respective section. Therefore, each is positive.

(3) Positive moment, **M**, tends to deform the component into a dish-shaped configuration such that it would hold water. Again, note that the moment acts in opposite directions on either side of the cut; nevertheless, each of the two moments tends to form a dish of its respective section. Therefore, each is positive.

Pay careful attention to Fig. 6.1.1 (b), (c), and (d) to see how positive internal forces act on opposite ends of a section isolated by two cuts.

Also, note that the sign convention for internal forces is not necessarily consistent with that for external forces. Because we will be calculating both internal and external forces in this section, be careful and apply the appropriate sign convention.

6.2 - Calculating internal forces.

In general, internal forces vary along the length of a structural component. Thus we must determine internal force as a function of position. The procedure is as follows:

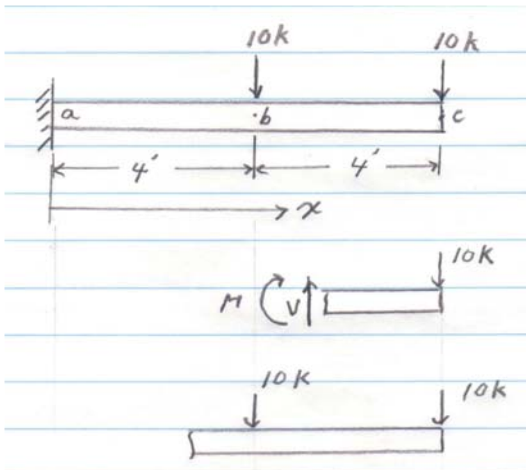
(1) Determine the reactions. Recall that for purposes of determining reactions, distributed loads can be replaced by equivalent concentrated loads. Be cautioned, however, that the original distributed loads must be restored before the <u>internal</u> forces are determined. More specifically, the original distributed loads must usually be restored when writing equations for internal forces within the section of the component on which the distributed loads act.

(2) Impose an imaginary section perpendicular to the axis of the component at the point where the internal forces are to be determined. Isolate the segment on one side of the cut and indicate the resultant of the unknown internal forces by drawing **N**, **V**, and **M** in their <u>positive</u> directions.

(3) Apply the equations of equilibrium ($\Sigma H= 0$, $\Sigma V = 0$, and $\Sigma M = 0$) to solve for **N**, **V**, and **M**. If any of these quantities comes out negative, then they act in the opposite direction from that originally assumed.

Generally speaking, there will be relative long sections of the member where the internal forces can be given as a simple function of the position coordinate, *x*. However, when a point is reached at which the type of loading changes or at which a concentrated load or moment is apply, the functional relationship will change. Thus it is usually only necessary to cut a section at a typical point within each portion of the component where loading is of a similar type.

Fig. 6.2.1 shows the procedure for determining the internal forces in a cantilever beam. Note that we do not need to solve for the reaction if we choose our segment to be to the right of the section that we cut. There are two ranges of *x* for which typical expressions must be developed: a to b, and b to c. Note carefully the notations that indicate the exact range of *x* values for which each of the two analysis steps is valid. Be particularly careful when dealing with the effect of point loads on shears. The shear just to the left of the load is different from the shear just to the right of the load. The same holds for moment diagrams at points where concentrated external moments are applied.



First, we will write equations valid for b < *x* ≤ c:

$$\Sigma V = 0 \Big|_{x>b}^{x\le c} \quad V_x - 10k = 0 \;\therefore\; V_x = 10k$$

$$\Sigma M = 0 \Big|_{x\ge b}^{x\le c} \quad M_x + 10k(8-x) = 0 \;\therefore\; M_x = -10k(8-x)$$

Note that the moment equation is actually valid at *x* = b because no external moment is applied at b. The shear equation, however, is valid only for *x* > b.

Next, we will write equations valid for a < *x* ≤ b:

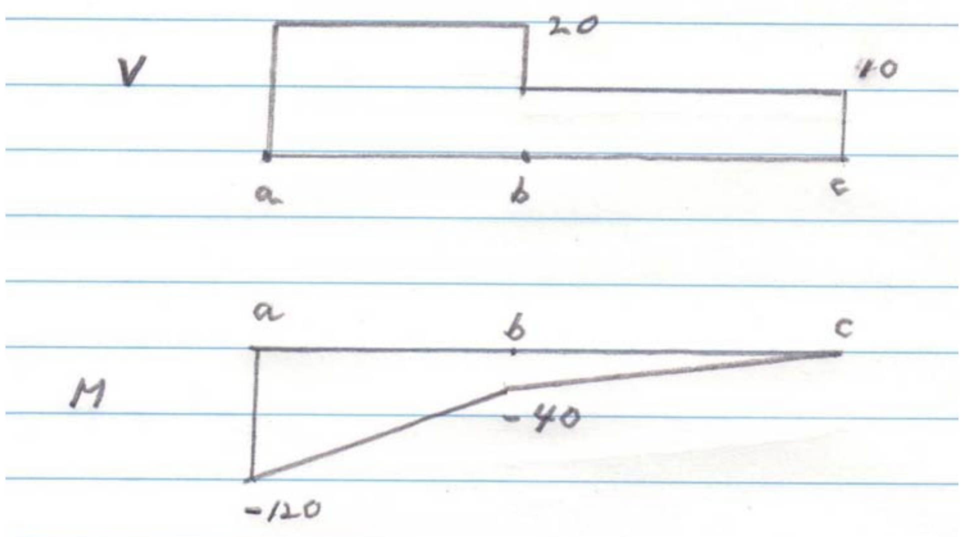
$$\sum V = 0 \Big|_{x>a}^{x \le b} \quad V_x - 10k - 10k = 0 \;\therefore\; V_x = 20k$$

$$\sum M = 0 \Big|_{x>a}^{x \le b} \quad M_x + 10k(8-x) + 10k(4-x) = 0$$

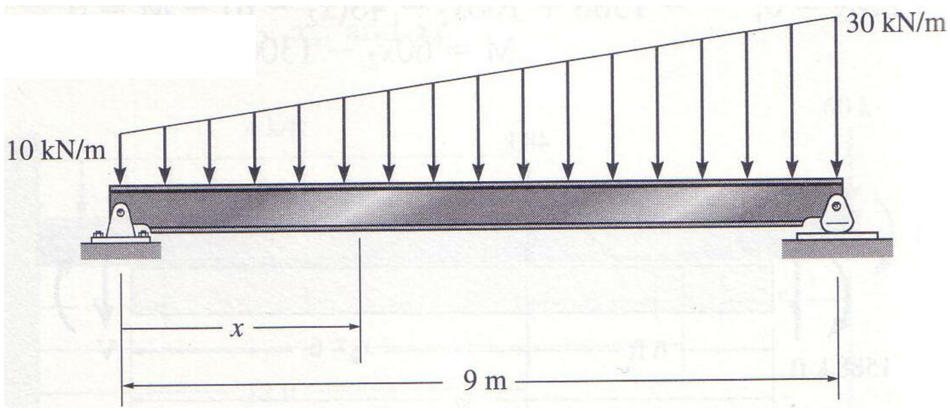
$$\therefore M_x = -120k + 20kx$$

Note that neither of these equations is valid for x = a because the reactions in effect apply an external moment and shear at a.

We can now plot the moment and shear along the beam:



As a second example, we will look at a distributed load acting on the simply supported beam shown below. The reactions are V_{left} = 75kN, V_{right} = 105kN, and H_{left} = 0.



Because the form of the loading does not change anywhere along the beam, single equations will suffice for moment and shear:

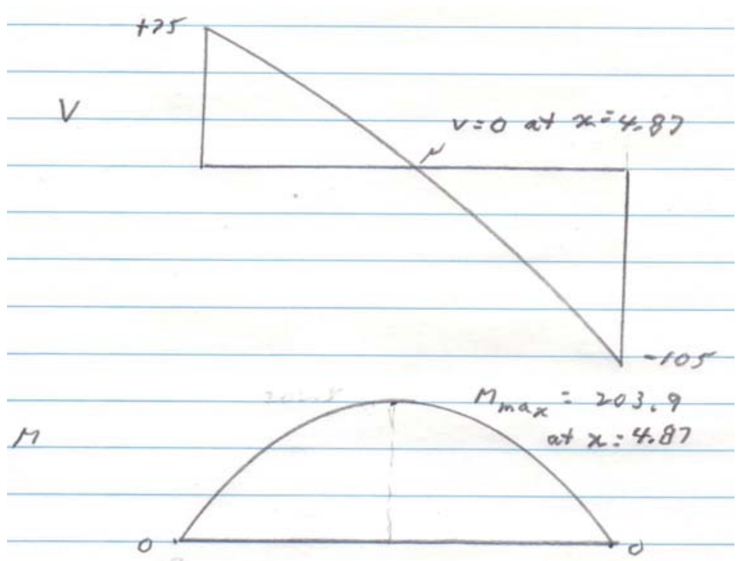
$$\Sigma V = 0 \overset{+}{\uparrow} \quad 75 - 10x - \frac{1}{2}(20)\left(\frac{x}{9}\right)x - V = 0$$

$$\therefore V = 75 - 10x - \frac{10}{9}x^2$$

$$\Sigma M = 0 \circlearrowleft \quad 75x - \left[(10x)\right]\frac{x}{2} - \left[\frac{1}{2}(20)\left(\frac{x}{9}\right)x\right]\frac{x}{3} - M = 0$$

$$M = 75x - 5x^2 - \frac{10}{27}x^3$$

Shear and moment are plotted below:



6.3 – Integral and differential relationships for internal forces. A set of integral and differential relationship can be derived for calculating shear from loads and for calculating moments from shears. In fact, a more complete integral relationship that requires principles that are beyond the scope of this course can also be used to calculated slopes and deflections of structures. For the sake of completeness, all of these relationships are shown in Fig. 6.3.1 below. The supporting mathematics is shown in the example problem of Example 6.3.1 below.

component	**integral relationships**	**differential relationships**
load, w	$w = \frac{dV}{dx}$	w, P, M
shear, V	$V = \frac{dM}{dx}$	$V = \int w dx + \sum P$
moment, M	M	$M = \int V dx + \sum M$
M/EI	$\frac{M}{\mathrm{EI}} = \frac{d\theta}{dx}$	$\frac{M}{\mathrm{EI}}$
slope, θ	$\theta = \frac{dy}{dx}$	$\theta = \int \frac{M}{EI} dx + C_1$
deflection, y	y	$y = \int \theta \, dx + C_2$

simple example

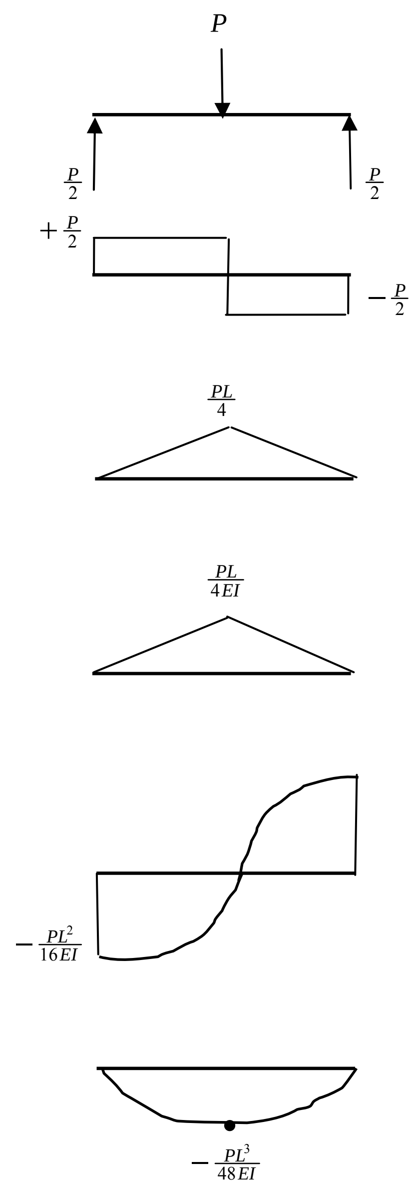


Fig. 6.3.1 – Integral and differential relationships for loads, shear, moments, slopes, and deflections.

Example 6.3.1 – Simply supported beam of length, L, with load, P, at mid-span.

Shear: $0 \le x < \frac{L}{2}$

$$V = \int w^{0}\, dx + \Sigma P = +\frac{P}{2}$$

$\frac{L}{2} \le x \le L$

$$V = \int w\, dx + \Sigma P = \frac{P}{2} - P = -\frac{P}{2}$$

Moment: $0 \le x \le \frac{L}{2}$

$$M = \int V\, dx + M^{0} = \int_{0}^{x} \frac{P}{2}\, dx = \frac{Px}{2}$$

$\frac{L}{2} \le x \le L$

$$M = \int_{0}^{L/2} \frac{P}{2}\, dx + \int_{L/2}^{x} -\frac{P}{2}\, dx$$

$$= \frac{PL}{4} + \frac{Px}{2} + \frac{PL}{4} = \frac{P}{2}(L - x)$$

$$M @ \frac{L}{2} = \frac{PL}{4}$$

Slope: $0 \le x \le \frac{L}{2}$

$$\Theta = \int \frac{M}{EI}\, dx + C_1 = \int \frac{Px}{2EI}\, dx + C_1 = \frac{Px^2}{4EI} + C_1$$

slope $= 0$ at $x = \frac{L}{2}$

$$\frac{PL^2}{16EI} + C_1 = 0 \qquad C_1 = -\frac{PL^2}{16EI}$$

$$\Theta = \frac{Px^2}{4EI} - \frac{PL^2}{16EI}$$

$\frac{L}{2} \le x \le L$

$$\Theta = \int \frac{M}{EI}\, dx + C_1' = \int \frac{P}{2EI}(L - x)\, dx + C_1'$$

$$= \frac{PLx}{2EI} - \frac{Px^2}{4EI} + C_1'$$

at $x = \frac{L}{2}$ $\theta = 0$

$$\frac{PL^2}{4EI} - \frac{PL^2}{16EI} + C_1' = 0$$

$$C_1' = \frac{PL^2}{16EI} - \frac{PL^2}{4EI} = -\frac{3PL^2}{16}$$

$$\theta = \frac{PLx}{2EI} - \frac{Px^2}{4EI} - \frac{3PL^2}{16EI}$$

at $x = L$

$$\theta = -\frac{PL^2}{2EI} + \frac{PL^2}{4EI} - \frac{3PL^2}{16EI} = \frac{PL^2}{16EI} \quad \text{ok}$$

at $x = L/2$

$$\theta = \frac{PL^2}{4EI} - \frac{PL^2}{16EI} - \frac{3PL^2}{16EI} = 0 \quad \text{ok}$$

deflection:

$0 \le x \le \frac{L}{2}$

$$y = \int \theta \, dx + C_2 = \int \left(\frac{Px^2}{4EI} - \frac{PL^2}{16EI} \right) dx + C_2$$

$$= \frac{Px^3}{12EI} - \frac{PL^2 x}{16EI} + C_2$$

at $x = 0$ $y = 0$

$C_2 = 0$

$$y = \frac{Px^3}{12EI} - \frac{PL^2 x}{16EI}$$

At $x = \frac{L}{2}$ $y = \frac{PL^3}{96EI} - \frac{PL^3}{32EI} = -\frac{PL^3}{48EI}$

$\frac{L}{2} \le x \le L$

$$y = \int \left[-\frac{Px^2}{4EI} + \frac{PLx}{2EI} - \frac{3PL^2}{16EI} \right] dx + C_2'$$

Example 6.3.1 – (Concluded)

$$y = -\frac{Px^3}{12EI} + \frac{PLx^2}{4EI} - \frac{3PL^2x}{16EI} + C_2'$$

at $x = L$ $y = 0$

$$C_2' = \frac{PL^3}{12EI} - \frac{PL^3}{4EI} + \frac{3PL^3}{16EI} = \frac{PL^3}{48EI}$$

$$y = -\frac{Px^3}{12EI} + \frac{PLx^2}{4EI} - \frac{3PL^2x}{16EI} + \frac{PL^3}{48EI}$$

at $x = L$

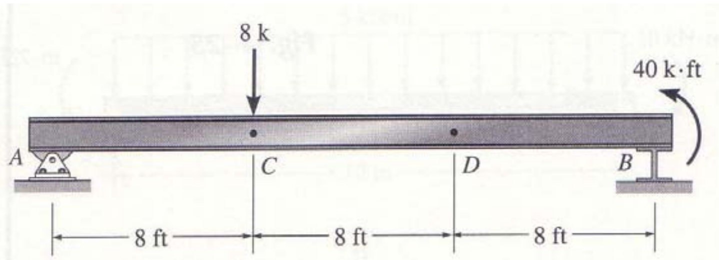
$$y = -\frac{PL^3}{12EI} + \frac{PL^3}{4EI} - \frac{3PL^3}{16EI} + \frac{PL^3}{48EI} = 0 \quad \text{OK}$$

at $x = L/2$

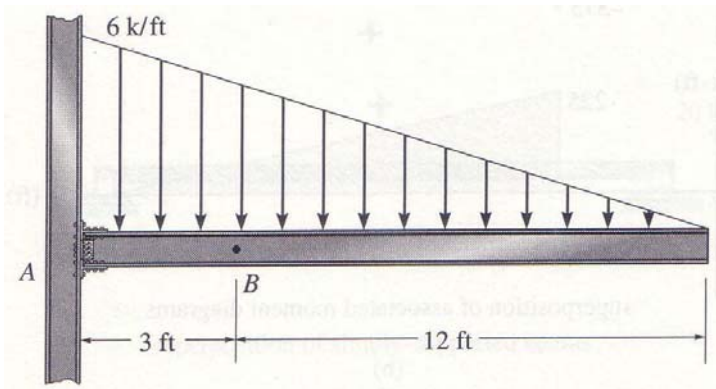
$$y = -\frac{PL^3}{96EI} + \frac{PL^3}{16EI} - \frac{3PL^3}{32EI} + \frac{PL^3}{48EI} = -\frac{PL^3}{48EI}$$

6.4 - Exercises.

Develop equations for shear and moment as a function of position for the following structural components. Plot the functions.



(a)



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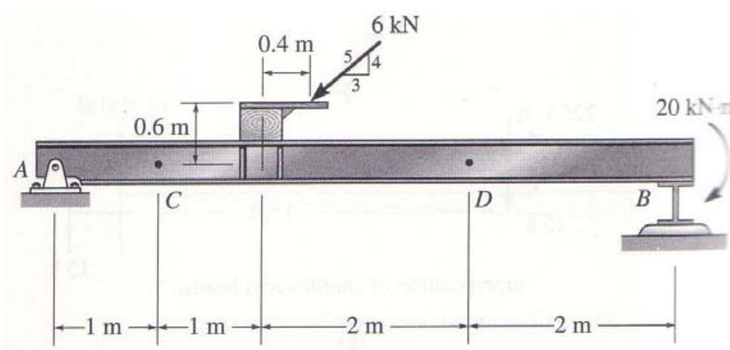

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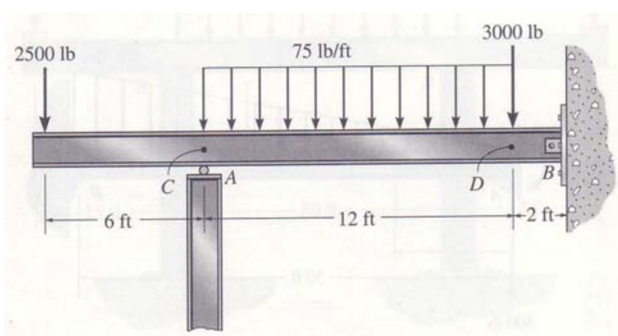

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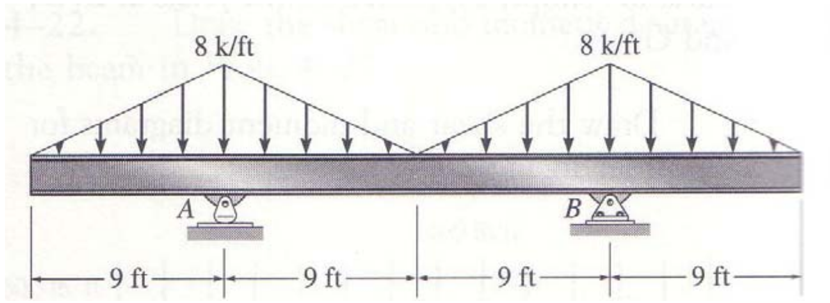

(e)